The boundary layer on a flat plate in a stream with uniform shear

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The incompressible laminar boundary layer on a semi-infinite flat plate is considered, when the main stream has uniform shear. A solution is obtained for the first two terms of an asymptotic solution for small viscosity. It is shown that one of the principal effects of free-stream vorticity is to introduce a modified pressure field outside the boundary-layer region.

1. Introduction

The region behind a bow shock wave is associated with an inviscid rotational flow. This vorticity in the inviscid flow will in some way affect the boundary layer on the body. To provide some understanding of the effects on the viscous boundary layer of a free-stream vorticity, the laminar boundary layer in a two-dimensional incompressible fluid of constant properties on a semi-infinite flat plate is studied, when the main stream has uniform shear.

Li (1955, 1956) and Yen (1955) have considered this problem. Li (1955) obtained a solution on the assumption that there is no pressure gradient as $y/\nu^{\frac{1}{2}} \to \infty$, where y is the Cartesian co-ordinate measured from the plate, taken as the x-axis, and ν is the kinematic viscosity. Yen used the same boundary condition but employed a Polhausen technique on the velocity profile in the boundary layer at different stations on the plate, and showed that a form factor is required, which depends on the boundary-layer thickness. Li (1956) retracted his first solution and included in his boundary conditions a pressure gradient at the edge of the boundary layer. Glauert (1957) used the same equations and, in effect, the same boundary conditions as Li (1955). He stated that the pressure gradient at the edge of the boundary layer must be zero. This pressure-gradient condition is considered below.

In this note a solution is obtained for the first two terms in an asymptotic solution for small viscosity. The effect of the shear on the boundary layer is $O(\nu^{\frac{1}{2}})$, which may be deduced from the introduction of a vorticity number—the ratio of the main-stream vorticity to the average vorticity in the boundary layer—or from perturbation considerations on the Navier–Stokes equation. Accordingly, the correction to the Blasius solution for a uniform main stream must be included to $O(\nu^{\frac{1}{2}})$. From consideration of the various boundary conditions and a study of the pressure gradient over the complete field of flow, a solution is obtained which is correct to $O(\nu^{\frac{1}{2}})$ everywhere in the field of flow. The solution

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being the first two terms of an *asymptotic* expansion, terms of $O(\nu)$ must be considered if they arise. The term of $O(\nu)$ which does appear is shown to be consistent with the modified definition of the displacement thickness which is necessary when the main-stream shear is present.

The difference in the following solution consists mathematically in the different treatment of the boundary conditions as $y/\nu^{\frac{1}{2}} \to \infty$ and in the method of solution.

2. Basic equations and solution

Let (x_2, y_2) be the rectangular Cartesian co-ordinates, with the origin at the leading edge, the x_2-axis along the plate and y_2 perpendicular to it. The velocity in the main stream is taken as $U_2 = U_0 + \Omega_0 y_2$. Take any length l as reference length. (Alternatively ν/U_0 or U_0/Ω_0 may be used.) Use non-dimensional co-ordinates and velocities denoted by subscripts 1 by taking all lengths as multiples of l, and velocities as multiples of U_0. From the continuity equation there is a non-dimensional stream-function ψ_1, such that $u_1 = \partial\psi_1/\partial y_1$, $v_1 = -\partial\psi_1/\partial x_1$. The dimensional stream-function is $U_0 l \psi_1$. The non-dimensional free-stream velocity is

$$U_1 = 1 + Ny_1, \quad N = \Omega_0 l/U_0. \tag{1 a, b}$$

The usual boundary-layer transformation is to stretch the y_1 co-ordinate by $y = R^{\frac{1}{2}} y_1$, and to use

$$\chi = R^{\frac{1}{2}}\psi_1 \tag{2}$$

as dependent variable, where R is the Reynolds number $U_0 l/\nu$. However, for a uniform main stream, the flow along the plate is obtained correctly to $O(R^{-\frac{1}{2}})$ for both the boundary-layer flow and the external stream if parabolic co-ordinates (ξ_1, η_1) are used. (For a discussion of optimal co-ordinates in general, see Kaplun (1954).) We therefore use parabolic co-ordinates here defined by

$$(\xi_1 + i\eta_1)^2 = x_1 + iy_1 \quad (\eta_1 \geqslant 0)$$

in the whole (x_1, y_1)-plane, with $\eta_1 = 0$ on the plate, $\xi_1 > 0$ on the upper half plane and $\xi_1 < 0$ on the lower half plane. We then stretch the η_1 co-ordinate by writing $\eta = R^{\frac{1}{2}}\eta_1$, and transform the stream-function by (2) as before. We also drop the subscript 1 from ξ_1. All the derivatives of χ with respect to ξ and η are bounded as $R \to \infty$, and correctly to $O(R^{-\frac{1}{2}})$ the equation for χ (boundary-layer equation) is

$$\frac{\partial}{\partial \eta}\left[\xi\left(\frac{\partial^3\chi}{\partial\eta^3} + \frac{\partial\chi}{\partial\xi}\frac{\partial^2\chi}{\partial\eta^2} - \frac{\partial\chi}{\partial\eta}\frac{\partial^2\chi}{\partial\xi\,\partial\eta}\right) + \left(\frac{\partial\chi}{\partial\eta}\right)^2\right] = 0. \tag{3}$$

For a uniform main stream, we have $N = 0$, and the solution in parabolic co-ordinates is then known to be

$$\chi = \xi f_0(\eta), \tag{4}$$

where f_0 is the Blasius solution.

For a given non-zero N it appears that the simplest asymptotic expansion for χ will commence with the terms

$$\chi = \xi f_0(\eta) + \frac{4N}{R^{\frac{1}{2}}}\xi^2 f_1(\eta) + \ldots, \tag{5}$$

the rotational contribution being necessarily even in ξ. Substituting into (3) and equating coefficients of ξ^2 and ξ^3, we get

$$\frac{\partial}{\partial \eta}(f_0''' + f_0 f_0'') = 0, \tag{6a}$$

and

$$\frac{\partial}{\partial \eta}(f_1''' + f_0 f_1'' - f_0' f_1' + 2 f_0'' f_1) = 0, \tag{6b}$$

where the prime denotes differentation with respect to η. On the plate $u = v = 0$, and so $f_0(0) = f_0'(0) = 0, f_1(0) = f_1'(0) = 0$. Also, as $\eta \to \infty, f_0' \to 2$. The boundary condition on f_1 as $\eta \to \infty$ will be discussed later.

Integration of (6a) with the given boundary condition at infinity yields the Blasius equation. Equation (6b) is

$$f_1^{(4)} + f_0 f_1''' + f_0'' f_1' + 2 f_0''' f_1 = 0, \tag{7}$$

which on integration gives

$$f_1''' + f_0 f_1'' - f_0' f_1' + 2 f_0'' f_1 = A_1, \tag{8}$$

where A_1 is a constant. Note that f_1 above is the particular case $n = -1$ in the general consideration of uniform flow along a flat plate made by Goldstein (1960).

As regards the condition on f_1 as $\eta \to \infty$, what we require is that the vorticity ω should asymptotically differ from the vorticity Ω_0 of the given main stream by an exponentially small amount, and that (with this exponentially small change of vorticity neglected) the difference in the irrotational flow obtained from the asymptotic form of the boundary-layer flow and that from the given main stream should itself tend to zero as the unstretched co-ordinate $\eta_1 \to \infty$. (Note that $\eta_1 \to \infty$ at every point whose distance from the nearest point of the plate $\to \infty$.)

By this means we satisfy (i) the physical condition that the diffusion of vorticity produced at the solid surface should contribute only an exponentially small vorticity outside the boundary layer, (ii) the condition that the boundary-layer flow should merge smoothly into the main-stream flow which itself satisfies (i), and (iii) the condition that the perturbation of the main stream, due to such effects as the displacement thickness, should vanish at an infinite distance. As shown below, these conditions are sufficient (as in the uniform flow case) for a solution to be defined. It is, of course, necessary that (iv) any induced pressure gradient in the region outside must be bounded; but when (iii) is satisfied, so is (iv).

For small η, we have

$$f_0(\eta) = \frac{\alpha\eta^2}{2!} - \frac{\alpha^2\eta^5}{5!} + \dots, \tag{9}$$

and, for large η,

$$f_0(\eta) \sim 2\eta - \beta, \quad f_0''(\eta) \sim \gamma e^{-\lambda^2} \quad (\lambda = \eta - \tfrac{1}{2}\beta), \tag{10}$$

where γ, α ($= 1{\cdot}3282$) and β ($= 1{\cdot}7208$) are constants.

The two solutions of (7) with double zeros at the origin are given by

$$\left.\begin{aligned} y_2(\eta) &= \frac{\alpha\eta^2}{2!} - \frac{\alpha^2\eta^5}{5!} + 23\frac{\alpha^3\eta^8}{8!} + \ldots, \\ y_3(\eta) &= \frac{\alpha\eta^3}{3!} - 2\frac{\alpha^2\eta^6}{6!} + 9\frac{\alpha^3\eta^9}{9!} + \ldots, \end{aligned}\right\} \tag{11}$$

for small enough η, of which y_2 is a complementary function of (8) and y_3 a particular integral of (8) with $A_1 = \alpha$. When η (or λ) is large, asymptotic approximations to complementary functions of (8) are (i) constant, (ii) $E_{-1}(\eta)$, (iii) $H_{-1}(\eta)$, where E_{-1} and H_{-1} are related to the cylindrical parabolic functions (Whittaker & Watson 1927) and

$$\left.\begin{aligned} &E_{-1}(\eta) = \lambda^2(1+\tfrac{1}{2}\lambda^{-2}) = \eta^2 - \beta\eta + \tfrac{1}{2}(1+\tfrac{1}{2}\beta^2) \\ \text{and} \quad &H_{-1}(\eta) \sim e^{-\lambda^2}\lambda^{-3}(1-3\lambda^{-2}+\ldots). \end{aligned}\right\} \tag{12}$$

A fourth complementary function of (7), which is a particular integral of (8) with $A_1 = \alpha$, is asymptotically equal to $-\frac{1}{2}\alpha\eta$. Consequently, there must be constants a_2, b_2, c_2, a_3, b_3, c_3 such that

$$\left.\begin{aligned} &y_2 \sim a_2 + b_2E_{-1} + c_2H_{-1} \\ \text{and} \quad &y_3 \sim -\tfrac{1}{2}\alpha\eta + a_3 + b_3E_{-1} + c_3H_{-1}. \end{aligned}\right\} \tag{13}$$

These equations were solved on a Univac machine and the constants evaluated, the result being

$$a_2 = 1{\cdot}0315, \quad b_2 = 0{\cdot}8354, \quad a_3 = 0{\cdot}96530, \quad b_3 = 1{\cdot}13165.$$

A solution of (8) for $f_1(\eta)$ may be taken in the form $Ay_2 + By_3$, where A and B are constants to be determined. Thus,

$$\begin{aligned} f_1(\eta) &= Ay_2 + By_3 \\ &\sim (Aa_2 + Ba_3) - \tfrac{1}{2}B\alpha\eta + (Ab_2 + Bb_3)\,E_{-1} + (Ac_2 + Bc_3)\,O(e^{-\lambda^2}) \\ &= [(Aa_2 + Ba_3) + \tfrac{1}{2}(Ab_2 + Bb_3)\,(1+\tfrac{1}{2}\beta^2)] \\ &\quad - [(Ab_2 + Bb_3)\,\beta + \tfrac{1}{2}B\alpha]\,\eta + (Ab_2 + Bb_3)\,\eta^2 \\ &\quad + (Ac_2 + Bc_3)\,O(e^{-\lambda^2}), \end{aligned}$$

and

$$f_0(\eta) \sim 2\eta - \beta.$$

From (1) we have, in the main stream,

$$\chi = 2\xi_1\eta + \frac{2N}{R^{\frac{1}{2}}}\,\xi_1^2\eta^2, \tag{14}$$

and so A and B must be chosen to make the coefficient of η^2 have the value $\frac{1}{2}$, that is,

$$Ab_2 + Bb_3 = \tfrac{1}{2}. \tag{15}$$

From (5) we have, for large η,

$$\chi \sim (2\eta - \beta) + \frac{4N}{R^{\frac{1}{2}}}\,\xi^2(\tfrac{1}{2}\eta^2 + C\eta + D),$$

neglecting exponentially small terms, where

$$C = -\tfrac{1}{2}(\alpha B + \beta), \quad D = (Aa_2 + Ba_3) + \tfrac{1}{8}(2 + \beta^2).$$

Hence $$\psi_1 \sim (2\xi_1\eta_1 + 2N\xi_1^2\eta_1^2) - \frac{\beta\xi_1}{R^{\frac{1}{2}}} + \frac{4NC}{R^{\frac{1}{2}}}\xi_1^2\eta_1 + \frac{4ND}{R}\xi_1^2.$$

This is now the boundary condition for small η_1 of the inviscid flow problem (that is the solution of $\nabla^2\psi_1 = N$), which must merge smoothly into the main-stream flow with the diffusion of vorticity produced at the plate exponentially small. With these boundary conditions the solution is

$$\psi_1 = (2\xi_1\eta_1 + 2N\xi_1^2\eta_1^2) - \frac{\beta\xi_1}{R^{\frac{1}{2}}} + \frac{4NC}{R^{\frac{1}{2}}}\left(\xi_1^2\eta_1 - \frac{\eta_1^3}{3}\right) + \frac{4ND}{R}(\xi_1^2 - \eta_1^2). \tag{16}$$

This stream-function satisfies conditions (i), (ii) for *any* C and D. A and B are undetermined and may still be chosen to make C *or* D equal to zero.

η	$f_1(\eta)$	$f_1'(\eta)$	η	$f_1(\eta)$	$f_1'(\eta)$
0	0	0	2·4	4·62326	2·57583
0·2	0·06021	0·59050	2·6	5·15005	2·69786
0·4	0·23139	1·10870	2·8	5·70444	2·85023
0·6	0·49837	1·54698	3·0	6·29155	3·02377
0·8	0·84423	1·89611	3·2	6·91480	3·21037
1·0	1·25044	2·14999	3·4	7·57617	3·40417
1·2	1·69800	2·31087	3·6	8·27670	3·60154
1·4	2·16951	2·39299	3·8	9·01689	3·80052
1·6	2·65164	2·42199	4·0	9·79695	4·00015
1·8	3·13689	2·42963	4·2	10·61697	4·20004
2·0	3·62408	2·44577	4·4	11·47697	4·40000
2·2	4·11712	2·49144	4·5	11·92197	4·50000

TABLE 1

Since $$u_1 = \frac{1}{2(\xi_1^2+\eta_1^2)^{\frac{1}{2}}}\frac{\partial\psi_1}{\partial\eta_1}, \quad v_1 = \frac{-1}{2(\xi_1^2+\eta_1^2)^{\frac{1}{2}}}\frac{\partial\psi_1}{\partial\xi_1},$$

u_1, with ψ_1 given by (16), tends to infinity with η_1 unless $C = 0$. With this value of C and equation (15), we get

$$B = -\beta/\alpha, \quad A = (1 + 2(\beta/\alpha)\,b_3)/2b_2, \tag{17a}$$

and $$C = 0, \quad D = a_2(1+2(\beta/\alpha)\,b_3)/2b_2 - (\beta/\alpha)\,a_3 + \tfrac{1}{8}(2+\beta^2). \tag{17b}$$

With these relations, $A = 2{\cdot}3534$, $B = -1{\cdot}2955$ and $f_1(\eta)$ and $f_1'(\eta)$ are as shown in table 1, and figure 1 below. With C and D from (17a, b), condition (iii) is satisfied. Denoting the pressure gradient in the extended boundary-layer flow by $(P_1, P_2, 0)$, we have

$$P_1 = -\frac{1}{2(\xi_1^2+\eta_1^2)^{\frac{1}{2}}}\left(u_1\frac{\partial u_1}{\partial\xi_1} + v_1\frac{\partial v_1}{\partial\xi_1}\right) - Nv_1,$$

and $$P_2 = -\frac{1}{2(\xi_1^2+\eta_1^2)^{\frac{1}{2}}}\left(u_1\frac{\partial u_1}{\partial\eta_1} + v_1\frac{\partial v_1}{\partial\eta_1}\right) + Nu_1,$$

with u_1, v_1 from above. If C were not equal to zero, for large ξ_1 and η_1 respectively, we would have

$$\left.\begin{aligned} P_1 &\sim \frac{4N^2D}{R}, & P_2 &\sim 0, \\ P_1 &\sim 0, & P_2 &\sim -2(N^2C/R^{\frac{1}{2}})\,\eta_1, \end{aligned}\right\} \tag{18}$$

and, for ξ_1 not small and small η_1 (i.e. at the edge of the boundary layer),

$$\left.\begin{aligned} P_1 &\to -\frac{N}{2R^{\frac{1}{2}}}(2C+\beta)\frac{1}{\xi_1}+\frac{2N^2(2D-C^2)}{R}+O(R^{-\frac{3}{2}}), \\ P_2 &\to O(R^{-\frac{3}{2}}). \end{aligned}\right\} \tag{19}$$

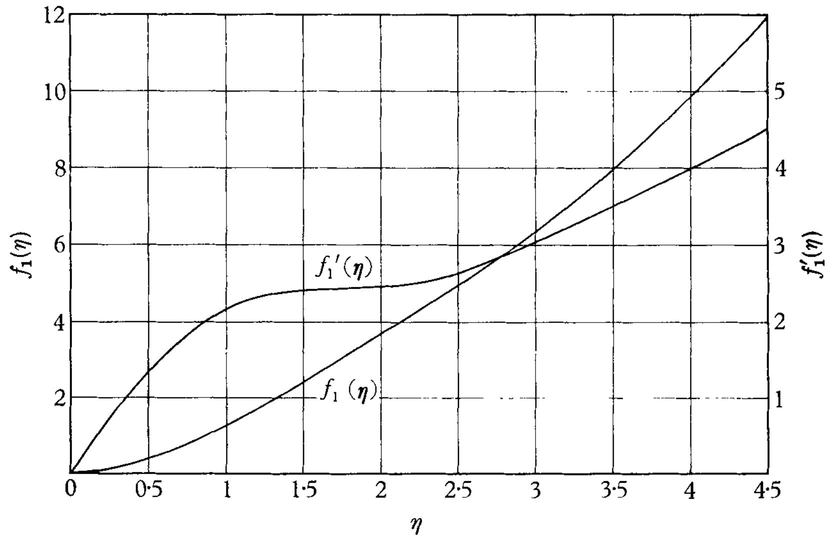

FIGURE 1

It may have been thought, from a consideration of the problem only as far from the plate as the edge of the boundary layer, that the pressure gradient (19) should be zero to $O(R^{-\frac{1}{2}})$, in which case C would be equal to $-\frac{1}{2}\beta$ (Glauert 1957). However, from (18), this would result in an unbounded pressure gradient in the free stream as $\eta_1 \to \infty$, which seems improbable from physical considerations. Therefore with $C = -\frac{1}{2}\beta$, although from (19) the pressure gradient is zero to $O(R^{-\frac{1}{2}})$, there is a pressure gradient growth as η_1 increases as in (18).

Thus, unless $C = 0$, condition (iii) (and (iv)) is violated. With C and D from (17*b*), equation (16) becomes

$$\psi_1 = (2\xi_1\eta_1 + 2N\xi_1^2\eta_1^2) - \frac{\beta\xi_1}{R^{\frac{1}{2}}} + \frac{4ND}{R}(\xi_1^2 - \eta_1^2). \tag{20}$$

The solution given by (5), and that for the inviscid region outside the boundary layer (that is, where the vorticity created at the plate is exponentially small) given by (20), then satisfies conditions (i) to (iii), and also (iv), to $O(R^{-\frac{1}{2}})$. From (19), with C and D from (17*b*), there is therefore an induced pressure gradient of $O(R^{-\frac{1}{2}})$ and equal to $-N\beta/2R^{\frac{1}{2}}\xi_1$ at the edge of the boundary layer.

The complete solution to the flow problem is given by (5) with A and B from (17a). The inviscid flow is given by (20) which gives the known Blasius solution for $N = 0$. Since we are seeking a solution, correct to $O(R^{-\frac{1}{2}})$, the term in (20) of $O(R^{-1})$ could perhaps be omitted. However, this term must be considered in relation to any continuation to $O(R^{-1})$ of the series solution. This term represents an outflow as $\eta_1 \to \infty$, which is consistent with the revised definition of the displacement thickness δ_1 when there is a main-stream shear present. (This has also been discussed by Curle (1957).) For ease of physical interpretation, we consider the definition in Cartesian co-ordinates in the form

$$\int_0^{y_1} (1+Ny_1)\,dy_1 = \int_0^{y_1+\delta_1} u_1\,dy_1, \tag{21}$$

where y_1 is the distance (outside the boundary layer) from the plate of the position at which the displacement thickness δ_1 is measured. This is the classical definition when $N = 0$. From (21), with u_1 obtained from (5) and (20), we get

$$\delta_1 = -\frac{1}{N}(1+Ny_1)+\frac{1}{N}\left[(1+Ny_1)^2+2N\left(\frac{\beta x_1^{\frac{1}{2}}}{R^{\frac{1}{2}}}-\frac{4ND}{R}x_1\right)\right]^{\frac{1}{2}}, \tag{22}$$

from which $\delta_1 \to 0$ as $y_1 \to \infty$, as it must since any finite displacement of the stream lines at infinity would correspond to an infinite increase in mass flow. In the case of uniform main-stream flow ($N = 0$), the inviscid-flow solution gives the displacement–thickness profile as the solid body for which it is the potential flow solution. From (21), if $y_1 = 0$ in the limits of the integrals in (22), we get at the edge of the boundary layer,

$$\delta_1 = -\frac{1}{N}+\frac{1}{N}\left[1+2N\left(\frac{\beta x_1^{\frac{1}{2}}}{R^{\frac{1}{2}}}-\frac{4ND}{R}x_1\right)\right]^{\frac{1}{2}}, \tag{23}$$

defined in terms of the defect of mass flux in the boundary layer, which is consistent with our definition. With δ_1 obtained from (23) as the profile of an equivalent body, equation (20) gives the inviscid flow past it, as in the case of a uniform main-stream flow. Care must be exercised in interpreting the result given by (23), since the correct defect of mass flux to $O(R^{-1})$ is not known completely until further terms are obtained in the asymptotic solution (5). It does, however, illustrate the consistency of the above solution and the definition of the displacement thickness. It is at this stage that a mathematical limitation on N is imposed, namely,

$$N < R^{\frac{1}{2}}/4Dx_1^{\frac{1}{2}}.$$

The shearing stress is given by $\tau_2 = \mu\,\partial u_2/\partial y_2$, and the contribution due to the free-stream vorticity is easily calculated from (5). In particular, according to (5), the skin friction on the plate is given in non-dimensional form by

$$\tau_0 = \left[\frac{\tau_2}{\rho U_0^2}\right]_{y_2=0} = \frac{f_0''(0)}{4R^{\frac{1}{2}}\xi_1}+\frac{Nf_1''(0)}{R}+\ldots, \tag{24}$$

where ρ is the density, and $\xi_1 = x_1^{\frac{1}{2}}$ on the plate. The second term in (24) gives the increase in the skin friction due to the main-stream vorticity. It is interesting to note that this contribution does not depend on the position ξ_1 on the plate. From (17*a*, *b*) and table 1, we get

$$\tau_0 = \frac{1}{R^{\frac{1}{2}} x_1^{\frac{1}{2}}} 0{\cdot}33206 + \frac{N}{R} 3{\cdot}1259 + \ldots. \tag{25}$$

Since $\xi_1 = x_1^{\frac{1}{2}}$ on the plate, outside the boundary layer, $\tau = N/R$, and so, from (23), the presence of the boundary layer increases the effect on the plate of the vorticity in the main stream. The variation with η of $f_1(\eta), f_1'(\eta)$ is shown in figure 1.

In conclusion, it may be said that one of the principal effects of free-stream vorticity is to introduce a modified pressure field outside the boundary-layer region. As a result, the skin friction, boundary-layer separation, and stability will all be affected. These effects are important when the displacement effects become significant. It should be noted that the displacement effect due to the free-stream vorticity is of $O(R^{-1})$.

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